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Dipper–Donkin algebra as global symmetry of quantum chains

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Abstract. We analyse the role of GL_2, a quantum group constructed by Dipper and Donkin (1991 *Proc. London Math. Soc.* **63** 165–211), as a global symmetry for quantum chains, and show how to construct all possible Hamiltonians for four-state quantum chains with GL_2 global symmetry. In doing this, we search for and present all inner actions of GL_2 on the Clifford algebra $C(1, 3)$. We also introduce the corresponding operator algebras, invariants and Hamiltonians, explicitly.

1. Introduction

In the last few years, quasi-triangular Hopf algebras or quantum groups have attracted a lot of attention from physicists. One of the most interesting features is that such structures can be related to underlying symmetries on spaces where the coordinates are noncommutative. When promoting these coordinates to functions, it has been shown that it is possible to write down an action for such fields that, when added to the action of a commuting field, has a symmetry resembling supersymmetry. Quantum groups can help us to understand the transformations on such fields and the action invariances.

Symmetry has always played an important role in theoretical physics in helping to reduce a problem with many variables to a more tractable size. The basis of the method built on the Bethe ansatz is to diagonalize the Hamiltonian along with an infinite set of constants of motion. In some cases the ocurrence of this infinite set of constants of motion is related to the appearance of a new kind of symmetry, the quantum-group symmetry. This nourishes the hope that, by relaxing the demands usually made on the structure of a symmetry group, and allowing a wider class of quantum groups, one can benefit from symmetry considerations in new situations, where a symmetry in the traditional sense is simply not present.

We model physical systems where the variables at the lattice sites take values so that the operators acting on them are matrices of dimension 4×4 and complex entries.

For some subclass of conformal integrable systems (well known examples are given by minimal models, Wess–Zumino–Witten (WZW) models and the Liouville–Toda theory) the underlying symmetries are indeed known to be given by quantum groups. However, in spite of extensive studies, our understanding of the quantum-group symmetry in these theories is still somewhat incomplete. We also think that there are realizations of quantum-group symmetry in nature.

The adjoint inner actions studied here are also called spectrum-generating quantum groups. So, we are classifying all possible roles of GL_2 as a spectrum-generating algebra for $C(1, 3)$.

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These inner actions have been used as a gauge transformation $H \to B$ in quantum-group gauge theory. Here B is the space–time algebra and H the coordinate ring of the gauge group.

Quantum GL_n are unique because they are related with q-Schur algebras [2], hence Hecke algebras and the representation theory of finite general linear groups. In fact, the representations of quantum GL_n provide the connection between the classical theory of polynomial representations of infinite general linear groups and the representation theory of finite general linear groups in the non-describing characteristic case; obviously if we take q to be one, we are in the classical case. In this limit the representation theory of Dipper–Donkin quantum groups is equivalent to the representation theory of q-Schur algebras. From this follows that the importance of this paper is to gain an understanding of the q-Schur algebra as a global symmetry for quantum chains. We remark that the Dipper–Donkin quantum groups are not special cases of the well known Manin's construction. There is one fundamental difference: *the Dipper–Donkin quantum determinant is, in general, not central*.

Weyl and Clifford algebras are at the heart of quantum physics. The most useful of them are those endowed with definite transformation properties under the action of some symmetry group. The idea that quantum groups could generalize Lie groups in describing symmetries of quantum physical systems has attracted much interest in the past decade.

In this paper we study the inner action of the Dipper–Donkin quantum group on the $C(1, 3)$ algebra, namely the algebra generated by the Dirac matrices, as a testing ground for applications of quantum-group symmetries. We search for the corresponding operator and invariant algebras in order to have additional information to propose Hamiltonians for quantum chains with this global symmetry. Interpreting the quantum group as a gauge group, one would consider only the invariant elements as observables. The rest of the algebra would then be an algebra of unobservable fields, whose function in the theory is to describe operations changing the superselection sector (creating charge). We are also interested in some fundamental questions. Can a quantum chain have global symmetry given by a quantum group with no central but group-like determinant? What would be the meaning of this?

We address here the first question and study (as a particular case) four-state quantum chains. We are able to show all possible, nontrivial Hamiltonians for this system, with Dipper–Donkin global symmetry.

Having discovered all the Hamiltonians which are invariant under the Dipper–Donkin quantum group for four-state quantum chains, we think of these as systems whose energy eigenstates organize into GL_2 multiplets, with no energy splitting among members of the same multiplet. This is done in spite of the fact that the GL_2 determinant is group-like as it should be, but not central to the algebra, as is the case.

2. Dipper–Donkin algebra

The algebraic structure of Dipper–Donkin quantization GL_2 [1] is generated by four elements c_{ij}, $1 \leqslant i, j \leqslant 2$ with relations which are presented in figure 1.

Here we denote by arrows $x \to y$ the 'quantum spinors' (or generators of the quantum

GL_2 :

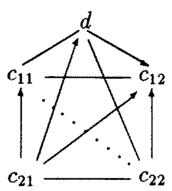

Figure 1. Diagramatic representation of the Dipper–Donkin algebra: 1.

plane [3]) $xy = qyx$. By the straight line $x - y$ we denote the 'classical spinors' $xy = yx$ and by dots $x \ldots y$ a classical spinor with a nontrivial *perturbation* [4], $xy - yx = p$ being $p = (q-1)c_{12}c_{21}$.

In this algebra the quantum determinant $d = c_{11}c_{22} - c_{12}c_{21}$ is noncentral and group-like. This is in contrast to Manin's approach [3]. A group-like element d, in a Hopf algebra, is such that $\Delta d = d \otimes d$ and $\epsilon(d) = 1$. In any Hopf algebra every group-like element is invertible, therefore the quantum GL_2 includes the formal inverse d^{-1}.

The coalgebra structure is defined in the standard way for all quantizations and the antipode S is given in [1].

As we already know, the Clifford algebra $C(1,3)$ is generated by the vectors γ_μ, $\mu = 0, 1, 2, 3$ with relations defined by the form $g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, as follows:

$$\begin{aligned}
&\gamma_\mu\gamma_\nu = g_{\mu\nu} + \gamma_{\mu\nu} \qquad \gamma_{\mu\nu} = -\gamma_{\nu\mu} \\
&\gamma_\rho\gamma_{\mu\nu} = g_{\rho\mu}\gamma_\nu - g_{\rho\nu}\gamma_\mu + \gamma_{\rho\mu\nu} \\
&\gamma_\lambda\gamma_{\mu\nu\rho} = g_{\lambda\mu}\gamma_{\nu\rho} - g_{\lambda\nu}\gamma_{\mu\rho} + g_{\lambda\rho}\gamma_{\mu\nu} + \gamma_{\lambda\mu\nu\rho}.
\end{aligned}$$

This algebra is isomorphic to the algebra of the 4×4 complex matrix and it includes the basis of matrix units reported in [5]: e_{ij}, $1 \leqslant i, j \leqslant 4$, among others.

An action of GL_2 on $C(1,3)$ is uniquely defined by the actions of c_{ij} on the generators of $C(1,3)$ [6,7];

$$c_{ij} \cdot \gamma_k = f_{ijk}(\gamma_0, \gamma_1, \gamma_2, \gamma_3) \tag{1}$$

where f_{ijk} are some noncommutative polynomials in four variables.

For every action $\cdot$ there exists an invertible matrix $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \in C(1,3)_{2\times 2}$, such that

$$c_{ij} \cdot v = \sum m_{ik} v m^*_{kj}$$

where $\begin{pmatrix} m^*_{11} & m^*_{12} \\ m^*_{11} & m^*_{12} \end{pmatrix} = M^{-1}$ (see the Skolem–Noether theorem for Hopf algebras [8,9]). The action $\cdot$ is called inner if the map $c_{ij} \to m_{ij}$ defines an algebra homorphism $\varphi : GL_2 \to C(1,3)$. Since the algebra $C(1,3)$ is isomorphic to the algebra of 4×4 matrices, the homorphism $C(1,3)$ defines (and is defined by) a four-dimensional module over (the algebraic structure of) GL_2, or, equivalently, a four-dimensional representation of GL_2.

If $\varphi(c_{12}c_{21}) = 0$, then by definition in figure 1 the representation φ is given for an essentially simpler structure, generated by two commuting 'quantum spinors' (c_{21}, c_{11}) and (c_{22}, c_{12}). In the case when $\varphi(c_{12}c_{21}) \neq 0$ we say that the inner action defined by φ has nonzero *perturbation*.

If we add the formal inverse c_{11}^{-1}, then the algebraic structure of Dipper–Donkin quantization GL_2 is generated by the elements in figure 2.

From here, it follows straightforwardly that, up to invertibility of c_{11}, the algebraic structure of GL_2 can be considered as a tensor product $\aleph \otimes \aleph$ where $\aleph$ is the quantum plane.

We say that the representation of the q-spinor $xy = qyx$, $x \to A$, $y \to B$ is *admissible* if there exists C such that $x \to C$, $y \to B$ and $x \to C$, $y \to A$ are also representations

GL_2 :

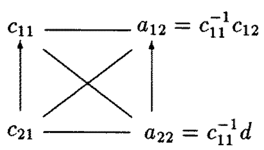

Figure 2. Diagramatic representation of Dipper–Donkin algebra: 2.

$$GL_q(2,C): \quad \begin{tikzcd} a_{11} \arrow[r] \arrow[d] \arrow[dr, dotted, dash] & a_{12} \arrow[d] \\ a_{21} \arrow[r] \arrow[ur, dash] & a_{22} \end{tikzcd}$$

Figure 3. Diagramatic representation of $GL_q(2, C)$.

of q-spinor with $CB \neq 0$. In other words, it means that $d \to A$, $c_{12} \to B$, $c_{21} \to C$ is a representation of the subalgebra of GL_2, generated by d, c_{12}, c_{21} with $CB \neq 0$.

Following the method already developed for studying the actions of $GL_q(2, C)$ on the Clifford algebra $C(1, 3)$ [5] we can construct all inner actions of the Dipper–Donkin quantization on this Clifford algebra, provided $q^m \neq 1$. *We can also can provide with the corresponding operator algebra* $\Re$ *(namely the image of the representation), the algebra of invariants* I *which is equal to the centralizer of* $\Re$ *in* $C(1, 3)$*, and the perturbation of the representation [4].* We define $c_{ij} \to C_{ij}$ to be a finite-dimensional representation of the quantum GL_2.

Here, we summarize the method used in [4, 5] to find and classify all possible inner actions of $GL_q(2, C)$ and GL_2 on $C(1, 3)$. The Hopf algebra $GL_q(2, C)$ is made out of q-spinors, classical spinors and kinds of perturbed spinors. The first and second types of spinors are defined as shown in this paper. For the perturbed spinors: in $GL_q(2, C)$, we consider $p = (q - q^{-1})a_{12}a_{21}$; meanwhile $p = (q - 1)c_{12}c_{21}$ for GL_2. Actually, $GL_q(2, C)$ can be presented as in figure 3: being a_{11}, a_{12}, a_{21}, a_{22}, d^{-1} generators of the algebra. Here $d = a_{11}a_{22} - qa_{12}a_{21}$. By arrows we again denote q-spinors, by the straight line commuting elements (classical spinors) and by dots, perturbed spinors.

For any Hopf algebra that can be presented in terms of combinations of q-spinors, classical spinors and perturbed spinors (whatever the definition of perturbed spinors is), we can use our method to find and fully classify all inner actions on any given algebra. So far we have used $C(1, 3)$ as a particular example on which $GL_q(2, C)$ and GL_2 are acting but we want to stress that our method allows us to use any other algebra. We choose $C(1, 3)$ since we want to have a realization isomorphic to the algebra of 4×4 complex matrices, having in mind further applications in quantum field theory. Moreover, in this paper some results are given in terms of matrix units.

First, we study all possible q-spinor representations on $C(1, 3)$ such that $q^3, q^4 \neq 1$ and analyse the equivalence of representations. Then, we simplify the algebraic structure of $GL_q(2, C)$ by defining an auxiliary algebra. From this, we find the representation of $SL_q(2, C)$ (provided $q^m \neq 1$) and *connected* $GL_q(2, C)$ [5].

Our classification scheme uses this *connection* and follows straightforwardly [5]. The operator algebras, the quantum determinants and the invariants of the corresponding inner actions can also be presented. Two of these given representations are equivalent if and only if they are equal to each other. Form this, we learn that, for $GL_q(2, C)$, the quantum determinants are the only quantum invariants.

The Dipper–Donkin algebra GL_2 is also generated by q-spinors, classical spinors and a perturbated spinor, like $GL_q(2, C)$. Our method can be applied with some extra conditions coming from the particular structure of GL_2 [4]. Following the steps reported above we find, for GL_2, that the corresponding algebra of invariants equals the centralizer of coefficients of M. By definition, the centralizer commutes with all elements of the algebra, thereby defining Hamiltonians with conserved energy.

Whenever the algebra of invariants is in the field (complex for our case) the Hamiltonian obtained is trivial. Besides, if and only if $I \in \Re$ the corresponding Hamiltonian can be defined, since this can be writen in terms of the generators for GL_2. In any other case the invariant algebra cannot be used to construct Hamiltonians of a quantum chain.

3. Quantum chains with global Dipper–Donkin symmetry

In this section we learn about global quantum-group symmetry. Although this does not lead straightforwardly to integrability, we hope to be able to provide some additional information related to this subject.

A method to construct quantum chains with symmetry associated to the algebra of functions on a particular quantum group has been recently used, for q as a root of unity [10].

In this section we address the role of the Dipper–Donkin quantum group as a global symmetry and show that, contrary to general belief (see e.g. [10]), the mixing of the generators in the coproduct of the corresponding coalgebra of a quantum group is not a sufficient condition to construct a nontrivial Hamiltonian for quantum chains. See, for instance, the constructions shown in [11–14].

A quantum chain with global quantum-group symmetry can be defined as follows [10, 11]: to each site $j = 1, \ldots, L$ of the chain, we assign a representation π_j. We write the Hamiltonian

$$H = \sum_{j=1}^{L-1} id \otimes \cdots \otimes id \otimes H_j \otimes id \otimes \cdots \otimes id$$

where H_j acts on sites j and $j+1$ as

$$H_j = (\pi_j \otimes \pi_{j+1})[Q_j(\Delta(C))].$$

Here j denotes the site of the lattice, C is a central element of the algebra, by π_j we mean a representation for the algebra and Q_j is a polynomial function.

First, let us study when q is a root of unity. In this case, this polynomial function can be taken of degree $d \leqslant p$ where the integer p is characterized by the value of q ($q^p = 1$) as is done in [10, 11].

If q is a root of unity ($q^p = 1$), all the elements c_{ij}^p are central. In this case, one can uniquely define a state $|0\rangle$ which is a common eigenvector of c_{22} and c_{21} with eigenvalues α and $\alpha\beta$ (α and β being arbitrary constants), respectively. Then we build the space V as the linear span of the vectors $|n\rangle = c_{12}^n|0\rangle$, $0 \leqslant n \leqslant p-1$. We can show that V is an invariant vector space under the action of the Dipper–Donkin quantum group. Thus, we construct π_j as follows:

$$\begin{array}{lll} c_{12}|n\rangle = |n+1\rangle \quad \text{for} \quad n < p-1 & & c_{22}|n\rangle = \alpha|n\rangle \\ c_{11}|n\rangle = \beta|n+1\rangle \quad \text{for} \quad n < p-1 & & c_{12}|p-1\rangle = \eta|0\rangle \\ c_{21}|n\rangle = q^n\alpha\beta|n\rangle \quad c_{11}|p-1\rangle = \beta\eta|0\rangle. & & \end{array} \tag{2}$$

Here η is the central value of c_{12}^p. All the parameters are independent.

We would like to remark that, in spite of the mixing of the generators in the coproduct for the Dipper–Donkin algebra which has the same structure as the coproduct defined for $GL_q(n)$, the Dipper–Donkin algebra leads only to trivial Hamiltonians (proportional to the identity). This is true even for two-state quantum chains for which it is known that a $GL_q(2)$ global symmetry can be implemented [10].

Now, let us study the case $q^m \neq 1$. We introduce here an alternative way to build up Hamiltonians with Dipper–Donkin quantum global symmetry. Since for any Dipper–Donkin quantum group the quantum determinant is group-like but not central and the invariants I are central, we can define a Hamiltonian as follows:

$$H_j = (\pi_j \otimes \pi_{j+1})[Q_j(\Delta(I)].$$

Our method works only for $q^m \neq 1$. Here Q_j is any polynomial function.

We propose, as a particular case, to study the Dipper–Donkin algebra like a global symmetry of four-state quantum chains. This is done by searching all possible finite-dimensional representations of the Dipper–Donkin group on the algebra of 4×4 complex matrices on which a well-defined coproduct for the algebra of invariants can be applied. Here π_j is one of these representations.

In table 1 we give the full set of all possible inner actions that, being nontrivial, are in the operator algebra $\Re$; thereby properly defining $\Delta(I)$ and a corresponding Hamiltonian. Each particular case provides a Hamiltonian with quantum Dipper–Donkin global symmetry for a four-state quantum chain. An important result of this paper is that for all the cases reported in table 1 *the Hamiltonians for four-state quantum chains with Dipper–Donkin global symmetry have the unique form* $H_j = (\pi_j \otimes \pi_{j+1})[Q_j(\Delta(A_j d + B_j C_{11} + C_j C_{22})]$ where A_j, B_j and C_j are constants also given in table 1. This, together with the representation of the GL_2 generators, straightforwardly leads to a Hamiltonian writen in matrix units, Dirac gamma matrices or 'mass' ($m_\pm = (1 \pm \gamma_0)/2$) and 'spin' ($s_{\uparrow/\downarrow} = (1 \pm \mathrm{i}\gamma_{12})/2$) operators.

Let us now introduce some concrete examples for each case (namely particular forms of the quantum determinant). To reach this goal we obtain at first $(\pi_j \otimes \pi_{j+1})[Q_j(\Delta(C_{11})]$. Explicitly,

$$\begin{aligned}(\pi_j \otimes \pi_{j+1})[Q_j(\Delta(C_{11} \otimes C_{11}) + Q_j(\Delta(C_{12} \otimes C_{21})] &= \pi_j[Q_j(C_{11}] \otimes \pi_{j+1}[Q_j(C_{11}] \\ &\quad + \pi_j[Q_j(C_{12}] \otimes \pi_{j+1}[Q_j(C_{21}].\end{aligned}$$

We consider the simplest case and take Q_j to be linear.

In a similar way we get $(\pi_j \otimes \pi_{j+1})[Q_j(\Delta(C_{22})]$. Explicitly

$$\begin{aligned}(\pi_j \otimes \pi_{j+1})[Q_j(\Delta(C_{21} \otimes C_{12}) + Q_j(\Delta(C_{22} \otimes C_{22})] &= \pi_j[Q_j(C_{21}] \otimes \pi_{j+1}[Q_j(C_{12}] \\ &\quad + \pi_j[Q_j(C_{22}] \otimes \pi_{j+1}[Q_j(C_{22}].\end{aligned}$$

Again, we consider the simplest case and take Q_j to be linear. At last, we obtain $(\pi_j \otimes \pi_{j+1})[Q_j(\Delta(d)]$. Explicitly,

$$\begin{aligned}(\pi_j \otimes \pi_{j+1})[Q_j(\Delta(d \otimes d)] &= \pi_j[Q_j(C_{11}C_{22})] \otimes \pi_{j+1}[Q_j(C_{11}C_{22})] - \pi_j[Q_j(C_{11}C_{22})] \\ &\quad \otimes \pi_{j+1}[Q_j(C_{12}C_{21})] - \pi_j[Q_j(C_{12}C_{21})] \otimes \pi_{j+1}[Q_j(C_{11}C_{22})] \\ &\quad + \pi_j[Q_j(C_{12}C_{21})] \otimes \pi_{j+1}[Q_j(C_{12}C_{21})].\end{aligned}$$

As usual, we consider Q_j to be linear.

We are now ready to present concrete examples.

Case 2.2.

$$H_j = (m_+ s_\uparrow + \alpha_j m_+ s_\downarrow + \beta_j m_- s_\uparrow + \gamma_j m_- s_\downarrow) \otimes (m_+ s_\uparrow + \alpha_{j+1} m_+ s_\downarrow + \beta_{j+1} m_- s_\uparrow + \gamma_{j+1} m_- s_\downarrow).$$

Case 3.5.

$$\begin{aligned}H_j &= (q^2/\alpha_j m_+ s_\uparrow + q^2/\alpha_j m_+ s_\downarrow + m_- s_\uparrow + m_- s_\downarrow - q^2/\alpha_j^2 m_+(\gamma_1 + \mathrm{i}\gamma_2)\gamma_3/2) \\ &\quad \otimes (q^2/\alpha_{j+1} m_+ s_\uparrow + q^2/\alpha_{j+1} m_+ s_\downarrow + m_- s_\uparrow + m_- s_\downarrow - q^2/\alpha_{j+1}^2 m_+(\gamma_1 + \mathrm{i}\gamma_2)\gamma_3/2).\end{aligned}$$

Case 4.4.

$$\begin{aligned}H_j &= A_j(\alpha_j m_+ s_\uparrow + q^2 m_+ s_\downarrow + q m_- s_\uparrow + m_- s_\downarrow) \otimes (\alpha_{j+1} m_+ s_\uparrow + q^2 m_+ s_\downarrow + q m_- s_\uparrow + m_- s_\downarrow) \\ &\quad + B_j(\delta_j m_+ s_\uparrow + q m_+ s_\downarrow + q m_- s_\uparrow + m_- s_\downarrow) \\ &\quad \otimes (\delta_{j+1} m_+ s_\uparrow + q m_+ s_\downarrow + q m_- s_\uparrow + m_- s_\downarrow) - B_j \gamma_j \beta_{j+1} m_+(\gamma_1 - \mathrm{i}\gamma_2)/2 \\ &\quad \otimes m_-(-\gamma_1 + \mathrm{i}\gamma_2)\gamma_3/2 + C_j(\alpha_j/\delta_j m_+ s_\uparrow + q m_+ s_\downarrow + m_- s_\uparrow + m_- s_\downarrow) \\ &\quad \otimes (\alpha_{j+1}/\delta_{j+1} m_+ s_\uparrow + q m_+ s_\downarrow + q m_- s_\uparrow + m_- s_\downarrow) \\ &\quad - C_j \gamma_{j+1} \beta_j m_-(-\gamma_1 + \mathrm{i}\gamma_2)\gamma_3/2 \otimes m_+(\gamma_1 - \mathrm{i}\gamma_2)/2.\end{aligned}$$

Table 1. GL_2 representations, corresponding operator algebras $\Re$, algebra of invariants which are used to define nontrivial Hamiltonians for four-state quantum chains and the coefficients in the unique expression for these Hamiltonians are presented. The classification used is given by means of the different determinants in GL_2. (*a*) Case 1: $d = \mathrm{diag}(q^2, q, 1, 1)$. (*b*) Case 2: $d = \mathrm{diag}(q^2, q, q, 1)$. (*c*) Case 3: $d = \mathrm{diag}(q^2, q^2, q^2, 1)$. (*d*) Case 4: $d = \mathrm{diag}(\alpha, q^2, q, 1)\alpha \neq 0, q^{-1}, 1, q, q^2, q^3$. (*e*) Case 5: $d = \mathrm{diag}(q^2, q^2, q, 1)+e_{12}$. (*f*) Case 6: $d = \mathrm{diag}(q^2, q, 1, 1)+e_{34}$.

(a)

Case 1.1	$C_{12} = \alpha e_{12} + \beta e_{24}$ $C_{21} = 0$ $C_{11} = \mathbf{1} + e_{34}$ $C_{22} = q^2 e_{11} + q e_{22} + e_{33} + e_{44} - e_{34}$	$\Re = \begin{pmatrix} * & * & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & \epsilon & * \\ 0 & 0 & 0 & \epsilon \end{pmatrix}$
dim $\Re$ 6 dim I 2	$I = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \gamma \\ 0 & 0 & 0 & \alpha \end{pmatrix}$	$A_j = C_j = 0$
Case 1.2	$C_{12} = 0$ $C_{21} = \alpha e_{21} + \beta e_{32}$ $C_{11} = e_{11} + q^{-1} e_{22} + q^{-2} e_{33} + q^{-2} e_{44} + e_{34}$ $C_{22} = q^2 \mathbf{1} - q^4 e_{34}$	$\Re = \begin{pmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ 0 & * & \epsilon & * \\ 0 & 0 & 0 & \epsilon \end{pmatrix}$
dim $\Re$ 6 dim I 2	$I = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & \alpha \end{pmatrix}$	$A_j = B_j = 0$
Case 1.3	$C_{12} = \alpha e_{12}$ $C_{21} = \beta e_{32}$ $C_{11} = e_{11} + e_{22} + q^{-1} e_{33} + q^{-1} e_{44} + e_{34}$ $C_{22} = q^2 e_{11} + q e_{22} + q e_{33} + q e_{44} - q^2 e_{34}$	$\Re = \begin{pmatrix} * & * & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & \epsilon & * \\ 0 & 0 & 0 & \epsilon \end{pmatrix}$
dim $\Re$ 6 dim I 2	$I = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & \alpha \end{pmatrix}$	$A_j = -C_j = -\alpha q^{-1}$ $B_j = \alpha$

(b)

Case 2.1	$C_{12} = q\lambda\delta e_{13}$ $C_{21} = \delta e_{43}$ $C_{11} = e_{11} + \alpha e_{22} + e_{33} + q^{-1} e_{44}$ $C_{22} = q^2 e_{11} + q\alpha^{-1} e_{22} + q e_{33} + q e_{44}$ $\alpha \neq 1$	$\Re = \begin{pmatrix} * & 0 & * & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & * & * \end{pmatrix}$
dim $\Re$ 6 dim I 2	$I = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$	$A_j = -C_j = \frac{\alpha(q^{-1}-1)}{q-1}$ $B_j = \alpha$
Case 2.2	$C_{12} = 0; \alpha \neq \beta \neq \gamma \neq 1$ $C_{21} = 0$ $C_{11} = e_{11} + \alpha e_{22} + \beta e_{33} + \gamma e_{44}$ $C_{22} = q^2 e_{11} + q\alpha^{-1} e_{22} + q\beta^{-1} e_{33} + \gamma^{-1} e_{44}$	$\Re = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$
dim $\Re$ 4 dim I 4	$I = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}$	$A_j = C_j = 0$
Case 2.3	$C_{12} = 0; \alpha \neq \beta$ $C_{21} = 0$ $C_{11} = e_{11} + \alpha e_{22} + \beta e_{33} + \gamma e_{44}$ $C_{22} = q^2 e_{11} + \frac{q}{\alpha} e_{22} + \frac{q}{\beta} e_{33} + \gamma^{-1} e_{44}$	$\Re = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$
dim $\Re$ 4 dim I 4	$I = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}$	$A_j = C_j = 0$

Table 1. (Continued)

Case 2.4	$C_{12}=0$ $C_{21}=0$ $C_{11}=e_{11}+\alpha e_{22}+\alpha e_{33}+\beta e_{44}+e_{23}$ $C_{22}=q^2e_{11}+\frac{q}{\alpha}e_{22}+\frac{q}{\alpha}e_{33}+\frac{1}{\beta}e_{44}-\frac{q}{\alpha^2}e_{23}$	$\Re=\begin{pmatrix}* & 0 & 0 & 0\\ 0 & \epsilon & * & 0\\ 0 & 0 & \epsilon & 0\\ 0 & 0 & 0 & *\end{pmatrix}$
dim $\Re$ 4 dim I 4	$I=\begin{pmatrix}\alpha & 0 & 0 & 0\\ 0 & \gamma & \delta & 0\\ 0 & 0 & \gamma & 0\\ 0 & 0 & 0 & \beta\end{pmatrix}$	$A_j=B_j=0$
Case 2.5	$C_{12}=q\lambda\delta e_{13}$ $C_{21}=\delta e_{43}$ $C_{11}=e_{11}+e_{22}+e_{33}+q^{-1}e_{44}+e_{23}$ $C_{22}=q^2e_{11}+qe_{22}+qe_{33}+qe_{44}-qe_{23}$	$\Re=\begin{pmatrix}* & 0 & * & 0\\ 0 & \epsilon & * & 0\\ 0 & 0 & \epsilon & 0\\ 0 & 0 & * & *\end{pmatrix}$
dim $\Re$ 6 dim I 2	$I=\begin{pmatrix}\alpha & 0 & 0 & 0\\ 0 & \alpha & \beta & 0\\ 0 & 0 & \alpha & 0\\ 0 & 0 & 0 & \alpha\end{pmatrix}$	$A_j=-C_j=\frac{\alpha(q^{-1}-1)}{q-1}$ $B_j=\alpha$
Case 2.6	$C_{12}=0$ $C_{21}=0$ $C_{11}=e_{11}+\alpha e_{22}+\alpha e_{33}+\beta e_{44}+e_{23}$ $C_{22}=q^2e_{11}+\frac{q}{\alpha}e_{22}+\frac{q}{\alpha}e_{33}+\frac{1}{\beta}e_{44}-\frac{q}{\alpha^2}e_{23}$	$\Re=\begin{pmatrix}* & 0 & 0 & 0\\ 0 & \epsilon & * & 0\\ 0 & 0 & \epsilon & 0\\ 0 & 0 & 0 & *\end{pmatrix}$
dim $\Re$ 4 dim I 3	$I=\begin{pmatrix}\alpha & 0 & 0 & 0\\ 0 & \beta & \gamma & 0\\ 0 & 0 & \beta & 0\\ 0 & 0 & 0 & \alpha\end{pmatrix}$	$A_j=B_j=0$
(c)		
Case 3.1	$C_{12}=\alpha e_{13}+\gamma e_{34}$ $C_{21}=0$ $C_{11}=\mathbf{1}+e_{12}$ $C_{22}=q^2e_{11}+q^2e_{22}+e_{33}+e_{44}$	$\Re=\begin{pmatrix}\epsilon & * & * & *\\ 0 & \epsilon & 0 & 0\\ 0 & 0 & * & *\\ 0 & 0 & 0 & *\end{pmatrix}$
dim $\Re$ 7 dim I 2	$I=\begin{pmatrix}\alpha & \beta & 0 & 0\\ 0 & \alpha & 0 & 0\\ 0 & 0 & \alpha & 0\\ 0 & 0 & 0 & \alpha\end{pmatrix}$	$A_j=C_j=0$
Case 3.2	$C_{12}=\gamma e_{34}$ $C_{21}=\beta e_{32}$ $C_{11}=qe_{11}+qe_{22}+e_{33}+e_{44}+e_{12}$ $C_{22}=qe_{11}+qe_{22}+qe_{33}+e_{44}-e_{12}$	$\Re=\begin{pmatrix}\epsilon & * & 0 & 0\\ 0 & \epsilon & 0 & 0\\ 0 & * & * & *\\ 0 & 0 & 0 & *\end{pmatrix}$
dim $\Re$ 6 dim I 2	$I=\begin{pmatrix}\alpha & \beta & 0 & 0\\ 0 & \alpha & 0 & 0\\ 0 & 0 & \alpha & 0\\ 0 & 0 & 0 & \alpha\end{pmatrix}$	$A_j=-C_j=-q^{-1}\alpha$ $B_j=\alpha$
Case 3.3	$C_{12}=\gamma e_{34}$ $C_{21}=0$ $C_{11}=\alpha e_{11}+\alpha e_{22}+e_{33}+e_{44}+e_{12}$ $C_{22}=\frac{q^2}{\alpha}e_{11}+\frac{q^2}{\alpha}e_{22}+qe_{33}+e_{44}-\frac{q^2}{\alpha^2}e_{12}$	$\Re=\begin{pmatrix}\epsilon & * & 0 & 0\\ 0 & \epsilon & 0 & 0\\ 0 & 0 & * & *\\ 0 & 0 & 0 & *\end{pmatrix}$
dim $\Re$ 5 dim I 3	$I=\begin{pmatrix}\beta & \gamma & 0 & 0\\ 0 & \beta & 0 & 0\\ 0 & 0 & \alpha & 0\\ 0 & 0 & 0 & \alpha\end{pmatrix}$	$A_j=C_j=0$
Case 3.4	$C_{12}=\beta e_{23}$ $C_{21}=\gamma e_{43}$ $C_{11}=\alpha e_{11}+qe_{22}+qe_{33}+e_{44}$ $C_{22}=\frac{q^2}{\alpha}e_{11}+qe_{22}+e_{33}+e_{44}$ $\alpha\neq q$	$\Re=\begin{pmatrix}* & 0 & 0 & 0\\ 0 & * & * & 0\\ 0 & 0 & * & 0\\ 0 & 0 & * & *\end{pmatrix}$

Table 1. (Continued)

dim $\Re$ 6 dim I 2	$I = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$	$A_j = -B_j = -q^{-1}\alpha$ $C_j = \alpha$
Case 3.5	$C_{12} = 0$ $C_{21} = \gamma e_{43}$ $C_{11} = \alpha e_{11} + \alpha e_{22} + q e_{33} + e_{44} + e_{12}$ $C_{22} = \frac{q^2}{\alpha} e_{11} + \frac{q^2}{\alpha} e_{22} + e_{33} + e_{44} - \frac{q^2}{\alpha^2} e_{12}$	$\Re = \begin{pmatrix} \epsilon & * & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & * & * \end{pmatrix}$
dim $\Re$ 5 dim I 3	$I = \begin{pmatrix} \beta & \varphi & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$	$A_j = B_j = 0$
Case 3.6	$C_{12} = \alpha e_{13}$ $C_{21} = \gamma e_{43}$ $C_{11} = q e_{11} + \alpha e_{22} + q e_{33} + e_{44}$ $C_{22} = q e_{11} + \frac{q^2}{\alpha} e_{22} + e_{33} + e_{44}$ $\alpha \neq q$	$\Re = \begin{pmatrix} * & 0 & * & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & * & * \end{pmatrix}$
dim $\Re$ 6 dim I 2	$I = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$	$A_j = -B_j = -q^{-1}\alpha$ $C_j = \alpha$
Case 3.7	$C_{12} = \alpha e_{13}$ $C_{21} = \gamma e_{43}$ $C_{11} = q e_{11} + q e_{22} + q e_{33} + e_{44} + e_{12}$ $C_{22} = q e_{11} + q e_{22} + e_{33} + e_{44} - e_{12}$	$\Re = \begin{pmatrix} \epsilon & * & * & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$
dim $\Re$ 6 dim I 2	$I = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$	$A_j = -C_j = -q^{-1}\alpha$ $B_j = \alpha$
(d)		
Case 4.1	$C_{12} = 0$ $C_{21} = \gamma e_{32} + \beta e_{43}$ $C_{11} = \delta e_{11} + q^2 e_{22} + q e_{33} + e_{44}$ $C_{22} = \frac{\alpha}{\delta} e_{11} + e_{22} + e_{33} + e_{44}$	$\Re = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \end{pmatrix}$
dim $\Re$ 6 dim I 2	$I = \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}$	$A_j = B_j = 0$
Case 4.2	$C_{12} = \beta e_{23} + \gamma e_{34}$ $C_{21} = 0$ $C_{11} = \delta e_{11} + e_{22} + e_{33} + e_{44}$ $C_{22} = \frac{\alpha}{\delta} e_{11} + q^2 e_{22} + q e_{33} + e_{44}$	$\Re = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$
dim $\Re$ 6 dim I 2	$I = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$	$A_j = C_j = 0$
Case 4.3	$C_{12} = \beta e_{34}$ $C_{21} = \gamma e_{32}$ $C_{11} = \delta e_{11} + q e_{22} + e_{33} + e_{44}$ $C_{22} = \frac{\alpha}{\delta} e_{11} + q e_{22} + q e_{33} + e_{44}$	$\Re = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$
dim $\Re$ 6 dim I 2	$I = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$	$A_j = -C_j = -q^{-1}\beta$ $B_j = \beta$

Table 1. (Continued)

Case 4.4	$C_{12} = \gamma e_{23}$ $C_{21} = \beta e_{43}$ $C_{11} = \delta e_{11} + q e_{22} + q e_{33} + e_{44}$ $C_{22} = \frac{\alpha}{\delta} e_{11} + q e_{22} + e_{33} + e_{44}$	$\mathfrak{R} = \begin{pmatrix} \ast & 0 & 0 & 0 \\ 0 & \ast & \ast & 0 \\ 0 & 0 & \ast & 0 \\ 0 & 0 & \ast & \ast \end{pmatrix}$
dim $\mathfrak{R}$ 6 dim I 2	$I = \begin{pmatrix} \gamma & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$	$A_j = -B_j = -q^{-1}\beta$ $C_j = \beta$
(e)		
Case 5.1	$C_{12} = 0$ $C_{21} = \alpha e_{43}$ $C_{11} = q^2\beta e_{11} + q^2\beta e_{22} + q\gamma e_{33} + \gamma e_{44} + \beta e_{12}$ $C_{22} = \beta^{-1} e_{11} + \beta^{-1} e_{22} + q\gamma^{-1} e_{33} + \gamma^{-1} e_{44}$	$\mathfrak{R} = \begin{pmatrix} \epsilon & \ast & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \ast & 0 \\ 0 & 0 & \ast & \ast \end{pmatrix}$
dim $\mathfrak{R}$ 5 dim I 3	$I = \begin{pmatrix} \beta & \gamma & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$	$A_j = -\gamma B_j - \gamma^{-1} C_j$
Case 5.2	$C_{12} = \alpha e_{13} + \beta e_{34}$ $C_{21} = 0$ $C_{11} = q^2 e_{11} + q^2 e_{22} + q^2 e_{33} + q^2 e_{44} + e_{12}$ $C_{22} = e_{11} + e_{22} + q^{-1} e_{33} + q^{-2} e_{44}$	$\mathfrak{R} = \begin{pmatrix} \epsilon & \ast & \ast & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \ast & \ast \\ 0 & 0 & 0 & \ast \end{pmatrix}$
dim $\mathfrak{R}$ 6 dim I 2	$I = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$	$A_j = C_j = 0$
Case 5.3	$C_{12} = \alpha e_{34}$ $C_{21} = \beta e_{32}$ $C_{11} = q^2 e_{11} + q^2 e_{22} + q e_{33} + q e_{44} + e_{12}$ $C_{22} = e_{11} + e_{22} + e_{33} + q^{-1} e_{44}$	$\mathfrak{R} = \begin{pmatrix} \epsilon & \ast & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & \ast & \ast & 0 \\ 0 & 0 & \ast & \ast \end{pmatrix}$
dim $\mathfrak{R}$ 6 dim I 2	$I = \begin{pmatrix} \alpha & \beta & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$	$A_j = -B_j = -q^{-1}\beta$ $C_j = \beta$
Case 5.4	$C_{12} = \alpha e_{34}$ $C_{21} = 0$ $C_{11} = q^2\beta e_{11} + q^2\beta e_{22} + \gamma e_{33} + \gamma e_{44} + \beta e_{12}$ $C_{22} = \beta^{-1} e_{11} + \beta^{-1} e_{22} + q\gamma^{-1} e_{33} + \gamma^{-1} e_{44}$	$\mathfrak{R} = \begin{pmatrix} \epsilon & \ast & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \ast & \ast \\ 0 & 0 & 0 & \ast \end{pmatrix}$
dim $\mathfrak{R}$ 5 dim I 3	$I = \begin{pmatrix} \alpha & \gamma & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$	$A_j = C_j = 0$
Case 5.5	$C_{12} = \alpha e_{13}$ $C_{21} = 0$ $C_{11} = q^2\beta e_{11} + q^2\beta e_{22} + q^2\beta e_{33} + \delta e_{44} + \beta e_{12}$ $C_{22} = \beta^{-1} e_{11} + \beta^{-1} e_{22} + q^{-1}\beta^{-1} e_{33} + \delta^{-1} e_{44}$	$\mathfrak{R} = \begin{pmatrix} \epsilon & \ast & \ast & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & \ast & 0 \\ 0 & 0 & 0 & \ast \end{pmatrix}$
dim $\mathfrak{R}$ 5 dim I 3	$I = \begin{pmatrix} \alpha & \gamma & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$	$A_j = C_j = 0$
Case 5.6	$C_{12} = 0$ $C_{21} = \alpha e_{32}$ $C_{11} = q^2\beta e_{11} + q^2\beta e_{22} + q\beta e_{33} + \delta e_{44} + \beta e_{12}$ $C_{22} = \beta^{-1} e_{11} + \beta^{-1} e_{22} + \beta^{-1} e_{33} + \delta^{-1} e_{44}$	$\mathfrak{R} = \begin{pmatrix} \epsilon & \ast & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & \ast & \ast & 0 \\ 0 & 0 & 0 & \ast \end{pmatrix}$

Table 1. (Continued)

dim $\Re$ 5 dim I 3	$I = \begin{pmatrix} \alpha & \gamma & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$	$A_j = -\beta B_j = -b\beta$
Case 5.7	$C_{12} = 0$ $C_{21} = 0$ $C_{11} = q^2\alpha e_{11} + q^2\alpha e_{22} + \beta e_{33} + \gamma e_{44} + \alpha e_{12}$ $C_{22} = \alpha^{-1} e_{11} + \alpha^{-1} e_{22} + q\beta^{-1} e_{33} + \gamma^{-1} e_{44}$	$\Re = \begin{pmatrix} \epsilon & * & 0 & 0 \\ 0 & \epsilon & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix}$
dim $\Re$ 4 dim I 4	$I = \begin{pmatrix} \gamma & \varphi & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$	$A_j = C_j = 0$
(f)		
Case 6.1	$C_{12} = 0$ $C_{21} = \alpha e_{21}$ $C_{11} = q\beta e_{11} + \beta e_{22} + \gamma e_{33} + \delta e_{44} + \delta e_{34}$ $C_{22} = q\beta^{-1} e_{11} + q\beta^{-1} e_{22} + \gamma^{-1} e_{33} + \delta^{-1} e_{44}$ either $\gamma = \delta = \beta$ or $\gamma = \delta = q\beta$ or $\gamma = \delta \neq \beta$ or $\gamma = \delta \neq q\beta$	$\Re = \begin{pmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & \epsilon & * \\ 0 & 0 & 0 & \epsilon \end{pmatrix}$
dim $\Re$ 5 dim I 3	$I = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & \gamma \\ 0 & 0 & 0 & \beta \end{pmatrix}$	$A_j = -q^{-1}\beta B_j$
Case 6.2	$C_{12} = 0$ $C_{21} = \alpha e_{21}$ $C_{11} = q\beta e_{11} + \beta e_{22} + \gamma e_{33} + \delta e_{44} + \delta e_{34}$ $C_{22} = q\beta^{-1} e_{11} + q\beta^{-1} e_{22} + \gamma^{-1} e_{33} + \delta^{-1} e_{44}$ neither $\gamma = \delta = \beta$ nor $\gamma = \delta = q\beta$ nor $\gamma = \delta \neq \beta$ nor $\gamma = \delta \neq q\beta$	$\Re = \begin{pmatrix} * & 0 & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$
dim $\Re$ 6 dim I 2	$I = \begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{pmatrix}$	$A_j = \frac{\delta\gamma^{-1}-1}{\gamma-\delta} C_j$ $B_j = \frac{\delta^{-1}-\gamma^{-1}}{\gamma-\delta} C_j$
Case 6.3	$C_{12} = \alpha e_{24} + \beta e_{12}$ $C_{21} = 0$ $C_{11} = \mathbf{1} + e_{34}$ $C_{22} = q^2 e_{11} + q e_{22} + e_{33} + e_{44}$	$\Re = \begin{pmatrix} * & * & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & \epsilon & * \\ 0 & 0 & 0 & \epsilon \end{pmatrix}$
dim $\Re$ 6 dim I 2	$I = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & \alpha \end{pmatrix}$	$A_j = C_j = 0$
Case 6.4	$C_{12} = \beta e_{12}$ $C_{21} = 0$ $C_{11} = \alpha e_{11} + \alpha e_{22} + \gamma e_{33} + \gamma e_{44} + \gamma e_{34}$ $C_{22} = \frac{q^2}{\alpha} e_{11} + \frac{q}{\alpha} e_{22} + \gamma^{-1} e_{33} + \gamma^{-1} e_{44}$	$\Re = \begin{pmatrix} * & * & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & \epsilon & * \\ 0 & 0 & 0 & \epsilon \end{pmatrix}$
dim $\Re$ 5 dim I 3	$I = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & \gamma \\ 0 & 0 & 0 & \beta \end{pmatrix}$	$A_j = C_j = 0$
Case 6.5	$C_{12} = \beta e_{12}$ $C_{21} = 0$ $C_{11} = \alpha e_{11} + \alpha e_{22} + \gamma e_{33} + \delta e_{44} + \delta e_{34}$ $C_{22} = \frac{q^2}{\alpha} e_{11} + \frac{q}{\alpha} e_{22} + \gamma^{-1} e_{33} + \delta^{-1} e_{44}$ $\gamma \neq \delta$	$\Re = \begin{pmatrix} * & * & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$

Table 1. (Continued)

dim $\Re$ 6 dim I 2	$I = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$	$A_j = -\gamma^{-1} C_j = 1$ $B_j = \frac{\delta^{-1}-\gamma^{-1}}{\gamma-\delta}$
Case 6.6	$C_{12} = \delta e_{24}$ $C_{21} = 0$ $C_{11} = \alpha e_{11} + \beta e_{22} + \gamma e_{33} + \beta e_{44} + \beta e_{34}$ $C_{22} = \frac{q^2}{\alpha} e_{11} + \frac{q}{\beta} e_{22} + \gamma^{-1} e_{33} + \beta^{-1} e_{44}$	$\Re = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & \epsilon & * \\ 0 & 0 & 0 & \epsilon \end{pmatrix}$
	either $\alpha = \beta = \gamma$ or $\gamma^{-1} = q^2 \alpha^{-1} = \beta^{-1}$	
dim $\Re$ 5 dim I 2	$I = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$	$B_j = -\beta^{-1} A_j$ $C_j = \frac{1-q}{q\beta^{-1}-\beta^{-1}} A_j$
Case 6.7	$C_{12} = \delta e_{24}$ $C_{21} = 0$ $C_{11} = \alpha e_{11} + \beta e_{22} + \gamma e_{33} + \beta e_{44} + \beta e_{34}$ $C_{22} = \frac{q^2}{\alpha} e_{11} + \frac{q}{\beta} e_{22} + \gamma^{-1} e_{33} + \beta^{-1} e_{44}$	$\Re = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$
	neither $\alpha = \beta = \gamma$ nor $1/\gamma = q^2/\alpha = 1/\beta$	
dim $\Re$ 6 dim I 2	$I = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$	$A_j = -\beta B_j$ $B_j = \beta^2 B_j$
Case 6.8	$C_{12} = 0$ $C_{21} = \beta e_{32}$ $C_{11} = \alpha e_{11} + q\gamma e_{22} + \gamma e_{33} + \delta e_{44} + \delta e_{34}$ $C_{22} = \frac{q^2}{\alpha} e_{11} + \gamma^{-1} e_{22} + \gamma^{-1} e_{33} + \delta^{-1} e_{44}$	$\Re = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & \epsilon & * \\ 0 & 0 & 0 & \epsilon \end{pmatrix}$
	either $\alpha = \gamma = \delta$ or $\alpha \neq \gamma \neq \delta$	
dim $\Re$ 5 dim I 3	$I = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & \gamma \\ 0 & 0 & 0 & \beta \end{pmatrix}$	For$\alpha \neq \gamma \neq \delta$ $A_j = \frac{1-\gamma\delta^{-1}}{\gamma-\delta} C_j = 1$ $B_j = \frac{\delta^{-1}-\gamma^{-1}}{\gamma-\delta}$ For$\alpha = \gamma = \delta$ $A_j = -\gamma B_j$
Case 6.9	$C_{12} = 0$ $C_{21} = \beta e_{32}$ $C_{11} = \alpha e_{11} + q\gamma e_{22} + \gamma e_{33} + \delta e_{44} + \delta e_{34}$ $C_{22} = \frac{q^2}{\alpha} e_{11} + \gamma^{-1} e_{22} + \gamma^{-1} e_{33} + \delta^{-1} e_{44}$	$\Re = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$
	neither $\alpha = \gamma = \delta$ nor $\alpha \neq \gamma \neq \delta$	
dim $\Re$ 6 dim I 2	$I = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}$	$A_j = B_j = 0$ $C_j = \frac{\delta^{-1}-\gamma^{-1}}{\gamma-\delta}$
Case 6.10	$C_{12} = 0$ $C_{21} = 0$ $C_{11} = \mathbf{1} + e_{34}$ $C_{22} = q^2 e_{11} + q e_{22} + e_{33} + e_{44}$	$\Re = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & \epsilon & * \\ 0 & 0 & 0 & \epsilon \end{pmatrix}$
dim $\Re$ 4 dim I 4	$I = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & \delta \\ 0 & 0 & 0 & \gamma \end{pmatrix}$	A_j, B_j, C_j arbitrary

Table 1. (Continued)

Case 6.11	$C_{12} = 0$ $C_{21} = 0$ $C_{11} = \alpha e_{11} + \beta e_{22} + \gamma e_{33} + \delta e_{44} + \delta e_{34}$ $C_{22} = \frac{q^2}{\alpha} e_{11} + \frac{q}{\beta} e_{22} + \gamma^{-1} e_{33} + \epsilon^{-1} e_{44}$ $\alpha \neq \beta \neq \gamma \neq \delta \neq \epsilon$	$\Re = \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$
dim $\Re$ 5 dim I 3	$I = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & \gamma \end{pmatrix}$	$B_j = C_j = \frac{\epsilon^{-1} - \gamma^{-1}}{\gamma - \delta}$ $A_j = \frac{\delta(\gamma^{-1} - \epsilon^{-1})}{\gamma - \delta}$

Case 5.5.

$$
\begin{aligned}
H_j = {} & (q^2\beta_j m_+ s_\uparrow + q^2\beta_j m_+ s_\downarrow + q^2\beta_j m_- s_\uparrow + \delta_j m_- s_\downarrow + \beta_j m_+(\gamma_1 + \mathrm{i}\gamma_2)\gamma_3/2) \\
& \otimes(q^2\beta_{j+1} m_+ s_\uparrow + q^2\beta_{j+1} m_+ s_\downarrow + q^2\beta_{j+1} m_- s_\uparrow + \delta_{j+1} m_- s_\downarrow \\
& + \beta_{j+1} m_+(\gamma_1 + \mathrm{i}\gamma_2)\gamma_3/2).
\end{aligned}
$$

Case 6.10.

$$
\begin{aligned}
H_j = {} & A_j(q^2 m_+ s_\uparrow + q m_+ s_\downarrow + m_- s_\uparrow + m_- s_\downarrow + m_-(\gamma_1 + \mathrm{i}\gamma_2)\gamma_3/2) \\
& \otimes(q^2 m_+ s_\uparrow + q m_+ s_\downarrow + m_- s_\uparrow + m_- s_\downarrow) \\
& - B_j(1 + m_-(\gamma_1 + \mathrm{i}\gamma_2)\gamma_3/2 \otimes (1 + m_-(\gamma_1 + \mathrm{i}\gamma_2)\gamma_3/2 \\
& + C_j(q^2 m_+ s_\uparrow + q m_+ s_\downarrow + m_- s_\uparrow + m_- s_\downarrow) \\
& \otimes(q^2 m_+ s_\uparrow + q m_+ s_\downarrow + m_- s_\uparrow + m_- s_\downarrow).
\end{aligned}
$$

From these concrete examples it is straightforward to see that the Hamiltonians we have introduced are somehow related to the Ashkin–Teller model, written in terms of Ising spin; namely with each site i we associate two spins (in our case, $m_\pm$ and $s_{\uparrow/\downarrow}$).

Some additional standard symmetries can also be identified. For example, in case 3.5, there is an invariance under the transformation $s_\uparrow \to s_\downarrow$.

Finally, in table 1, we explicitly show all the inner actions of GL_2 on $C(1, 3)$ which can provide us with nontrivial Hamiltonians for four-state quantum chains with Dipper–Donkin quantum global symmetry, the operator algebra $\Re$, the algebra of invariants I and the value of the coefficients A_j, B_j and C_j in the unique expression for the corresponding Hamiltonians. In all reported cases the perturbation is zero.

4. Summary and conclusions

We have been able to show how to construct all possible Hamiltonians for four-state quantum chains with Dipper–Donkin global symmetry, for $q^m \neq 1$. This has been done, although the Dipper–Donkin quantum group has a noncentral but group-like determinant. We used the algebra of invariants for the actions of GL_2 on $C(1, 3)$ which corresponds to the centralizer of the operator algebra, or image of the representation.

It is straightforward to see that in all possible cases the perturbation of the corresponding action is zero. Moreover, there are only a few cases where all generators are not null. This occurs whenever the algebra of invariants is of the form $\alpha\mathbf{1} + \beta e_{ii+1}$ or $\alpha e_{ii} + \beta e_{jj}$ (being $(ii) = (11)$ and $(jj) = (22) = (33) = (44)$ or $(ii) = (22)$ and $(jj) = (11) = (33) = (44)$). We find out that all the Hamiltonians for four-state quantum chains with Dipper–Donkin global symmetry have the following unique form:

$$H = \sum_{j=1}^{L-1} id \otimes \ldots \otimes id \otimes (\pi_j \otimes \pi_{j+1})[Q_j(\Delta(A_j d + B_j C_{11} + C_j C_{22}))] \otimes id \ldots \otimes id$$

and we provide the reader with the specific values for A_j, B_j and C_j in all cases. Some concrete examples, written in terms of $m_{\pm}$ and $s_{\uparrow/\downarrow}$ are also introduced.

We report elsewhere [15] the complete classification of all inner actions of the Dipper–Donkin quantum group on the $C(1, 3)$ algebra. In that paper it can be seen how all invariants of the corresponding case 4 generate trivial Hamiltonians (this is not shown in the present paper), all invariants of the corresponding case 5 are $C \otimes C$ (this is case 4 in this paper), and finally all invariants of the corresponding case 6 (in this paper, case 5) are diagonal plus βe_{12}. It is also remarkable that all invariant algebras used to construct Hamiltonians with Dipper–Donkin global symmetry for four-state quantum chains are either diagonal or have elements in the diagonal plus βe_{ii+1}.

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